Lecture 11 **2012/12/10**

Outline

1. **Motivation**
2. **Hartman-Grobman Theorem**
3. **Center Manifold Theorems**
4. **Summary**

1. Motivation

- The Hartman-Grobman theorem is another very important result in the local qualitative theory of ODE. The theorem shows that $x' = f(x)$ with $f(0) = 0$ and its linearized system $x' = Df(0)\,x$ have the same qualitative structures near a hyperbolic equilibrium.
- The center manifold theorem is a natural extension of the stable manifold theorem, but it has an important difference. The proof is much complicated. This is an advanced topic in both qualitative analysis and stability theory.

2. Hartman-Grobman Theorem

1) Equivalence

Consider

$$x' = f(x), \tag{11.1}$$

where $x = 0$ is a hyperbolic equilibrium. The linearized system

$$x' = Ax, \tag{11.2}$$

where $A = Df(0)$.

Definition 11.1 Let A and B be subsets of R^n. A **homemorphism** A **onto** B is a **continuous one-to-one** map of A **onto** B, $H: A \to B$, such that $H^{-1}: B \to A$ is **continuous**. **The two sets** A and B are called **homemorphic** or **topologically equivalent** if there is a **homemorphism** from A **onto** B.

Remark 11.1 Notice the difference between **isomorphism** and **homemorphism.** The latter has the requirement of H having a continuous property.

Definition 11.2 Two dynamic systems such as (11.1) and (11.2) are said to be **topologically equivalent** in a neighborhood of the origin if there is a homemorphism H, mapping an open set U containing the origin onto an open set V containing the origin, which maps trajectories of (11.1) in U onto trajectories of (11.2) in V and preserves the orientation. If the homemorphism H preserves parameterization by time t, then (11.1) and (11.2) are said **topologically conjugate** in the neighborhood of the origin.

Remark 11.2 Notice the difference between **topologically equivalence** and **topologically conjugate.**

2) An Illustrative Example for the Topologically Conjugate

Example 11.1 Consider two linear systems $x' = Ax$ and $x' = Bx$ with

$$A = \begin{pmatrix} -1 & -3 \\ -3 & -1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & 0 \\ 0 & -4 \end{pmatrix}.$$

Let $H(x) = R\,x$, where

$$R = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \text{ and } R^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Then $B = R\,A\,R^{-1}$ and letting $y = H(x) = R\,x$ or $x = R^{-1}y$ gives

$$y = R\,A\,R^{-1}y = B\,y\,.$$

Thus, if $x(t) = e^{At}x_0$ is the solution of $x' = Ax$ through x_0, then

$$y(t) = H(x(t)) = R\,x(t) = R\,e^{At}x_0 = e^{Bt}R\,x_0$$

is the solution of $x' = Bx$; i.e. H maps trajectories of $x' = Ax$ onto trajectories of $x' = Bx$ and it preserves the parameterization by t since

$$H\,e^{At} = e^{Bt}H\,.$$

Therefore, H is a homemorphism from A onto B. $x' = Ax$ and $x' = Bx$ are topologically conjugate.

Remark 11.3 The mapping $H(x) = R\,x$ is in fact a rotation through 45^0 and it is clearly a homemorphism. See Fig. 11.1 as follows.

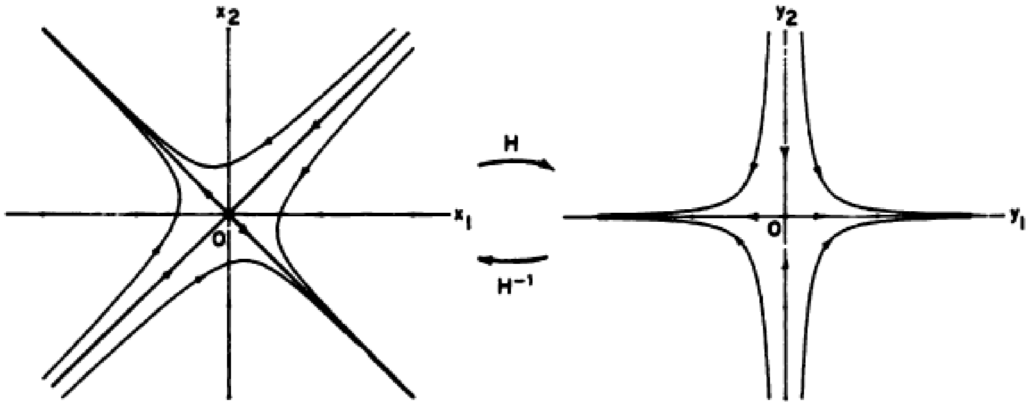

Fig. 11.1

3) Statement of Hartman-Grobman Theorem

Theorem 11.1 If $x = 0$ is a hyperbolic equilibrium of (11.1) and (11.2), then there exists a homemorphism H of an open set U containing the origin onto an open set V containing the origin such that for each $x_0 \in U$, there exists an open interval $I_0 \subset R$ containing the origin such that for all $t \in I_0$

$$H \circ \varphi_t(x_0) = e^{At} H(x_0).$$

Remark 11.4 The proof of Theorem 11.1 is not presented. One may consults the book of “Differential Equations and Dynamical Systems” 3rd ed. by Lawrence Perko at pp. 121-124.

4) An Example Showing Hartman-Grobman Theorem

Example 11.2 Consider

$$y' = -y; \quad z' = z + y^2. \tag{11.3}$$

The solution with $y(0) = y_0$ and $z(0) = z_0$ is solved by

$$y(t) = y_0 e^{-t}; \quad z(t) = z_0 e^t + \frac{y_0^2}{3}(e^t - e^{-2t}).$$

The linearized system is given by

$$y' = -y, \quad z' = z. \tag{11.4}$$

Its solution with $y(0) = y_0$ and $z(0) = z_0$ is easily solved by $y(t) = y_0 e^{-t}$,

$z(t) = z_0 e^t$. The homemorphism H is defined as $H(y,x) = \begin{pmatrix} y \\ z + \dfrac{y^2}{3} \end{pmatrix}$. Then, we verify the result of the Hartman-Grobman Theorem as follows. Let the solution of the original system as

$$\varphi_t(y_0, z_0) = \begin{pmatrix} y_0 e^{-t} \\ z_0 e^t + \dfrac{y_0^2}{3}(e^t - e^{-2t}) \end{pmatrix}$$

and e^{At} of the linearized system is given by $e^{At} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix}$. Since

$$e^{At} H(y_0, z_0) = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} \begin{pmatrix} y_0 \\ z_0 + \dfrac{y_0^2}{3} \end{pmatrix} = \begin{pmatrix} e^{-t} y_0 \\ e^t (z_0 + \dfrac{y_0^2}{3}) \end{pmatrix};$$

and

$$H \circ \varphi_t(y_0, z_0) = H \circ \begin{pmatrix} y_0 e^{-t} \\ z_0 e^t + \dfrac{y_0^2}{3}(e^t - e^{-2t}) \end{pmatrix} = \begin{pmatrix} y \\ z + \dfrac{y^2}{3} \end{pmatrix} \Bigg|_{\substack{y = y_0 e^{-t} \\ z = z_0 e^t + \frac{y_0^2}{3}(e^t - e^{-2t})}}$$

$$= \begin{pmatrix} y_0 e^{-t} \\ z_0 e^t + \dfrac{y_0^2}{3}(e^t - e^{-2t}) + \dfrac{(y_0 e^{-t})^2}{3} \end{pmatrix} = \begin{pmatrix} e^{-t} y_0 \\ e^t (z_0 + \dfrac{y_0^2}{3}) \end{pmatrix},$$

we have

$$H \circ \varphi_t(y_0, z_0) = e^{At} H(y_0, z_0) \quad \text{for all} \quad t \geq 0.$$

The nonlinear system (11.3) and the linearized system (11.4) of this example are shown in Fig. 11.2 as follows,

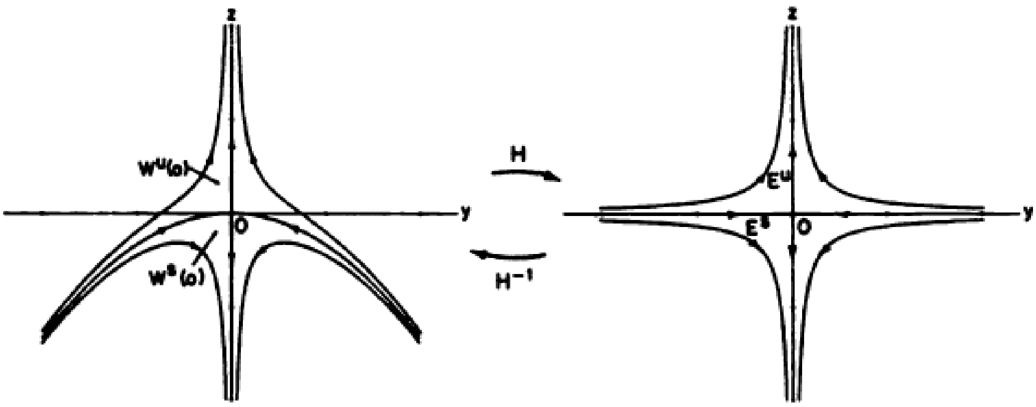

Fig. 11.2

where "$W^s(0)=\{(y,z)\in R^2 \mid z=-\frac{y^2}{3}\}$ maps onto $E^s=\{(y,z)\in R^2 \mid z=0\}$ and $W^u(0)=\{(y,z)\in R^2 \mid y=0\}$ maps onto $E^u=\{(y,z)\in R^2 \mid y=0\}$ by H.

Trajectories, such as $z=\frac{1}{y}-\frac{y^2}{3}$ of (11.3) maps onto trajectories, such as $z=\frac{1}{y}$ of (11.4), by H and H preserves the parameterization of t." **(The details of computation "$\cdots$" are left for homework)**

Remark 11.5 How to find a homemorphism H such that $H\circ\varphi_t(x_0)=e^{At}H(x_0)$ is difficult. In fact, the Hartman-Grobman Theorem only assures the existence of H. It doesn't tell us any information on how to find H**.** It is a qualitative property!

Remark 11.6 If f is of C^r, $r\geq 1$, then it can be proved that there exists a C^1-homemorphism H such that $H\circ\varphi_t(x_0)=e^{At}H(x_0)$ by Hartman in 1960. It should be noted that assuming higher derivatives of f doesn't imply the existence of higher derivatives of H**.** In general even if f is analytic, there does not exist a mapping H of class C^2 satisfying $H\circ\varphi_t(x_0)=e^{At}H(x_0)$. If the eigenvalues of $Df(0)$ satisfy some extra conditions, it may increase the smoothness of H**.** The discussion on this matter may consults the book of "Differential Equations and Dynamical Systems" by L. Perko, 3rd, Springer, pp. 127, 2001,.

3. Center Manifold Theorems

The stable manifold theorem gives a complete description of the dynamics of (11.1) in a neighborhood of a hyperbolic equilibrium. The center manifold theorem does provide such a description if we determine the behavior of solutions on the center manifold W^c.

1) The Linearized System with Zero Real Part

Consider (11.1) and (11.2). If A in (11.2) has k eigenvalues with zero real parts, j eigenvalues with positive real parts and $n-k-j$ eigenvalues with negative real parts, then $R^n = E^c \oplus E^u \oplus E^s$ with $\dim E^c = k$, $\dim E^u = j$ and $\dim E^s = n-k-j$. This situation can also be extended to (11.1) near $x=0$.

First we consider the case where E^u is trivial, i.e. A has no eigenvalues with positive real parts. In this case, $R^n = E^c \oplus E^s$ with $\dim E^c = k$ and $\dim E^s = n-k$.

2) Center Manifold Theorem

Theorem 11.2 Suppose that A in (11.2) has k eigenvalues with zero real parts and $n-k$ eigenvalues with negative real parts. Then, there exist

1. an k-dimensional center manifold $W^c(0)$ of class C^1 for (11.1) with $\dim W^c(0) = \dim E^c$, tangent to the center subspace E^c at $x=0$, which is invariant under the flow φ_t of (11.1);
2. an $n-k$-dimensional stable manifold $W^s(0)$ of class C^1 for (11.1) with $\dim W^s(0) = \dim E^s$, tangent to the stable subspace E^s at $x=0$, which is invariant under the flow φ_t of (11.1) and φ_t that starts on $W^s(0)$ is exponentially decay as $t \to +\infty$.

3) An illustrative Example with Many Local Center Manifolds

Consider the following typical system for showing many center manifolds.

$$x' = f(x),$$

where $f(x) = \begin{pmatrix} x_1^2 \\ -x_2 \end{pmatrix}$. The origin is only equilibrium. The linearized system is given by

$$x' = Df(0)\,x,$$

where $A = Df(0) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$. Obviously, the stable subspace E^s is x_2-axis and the center subspace E^c is x_1-axis. The solution with $x(0) = c = (c_1, c_2)^T$ is easily solved by

$$x_1(t) = \frac{c_1}{1 - c_1 t}; \quad x_2(t) = c_2 e^{-t}.$$

Eliminating t yields $x_2 = c_2 e^{-\frac{1}{c_1}} e^{\frac{1}{x_1}}$. Then, for any $c_1 < 0$, c_2, the function

$$x_2 = \psi(x_1, c_1, c_2) = \begin{cases} c_2 e^{-\frac{1}{c_1}} e^{\frac{1}{x_1}}, & x_1 < 0 \\ 0, & x_1 = 0 \end{cases} \tag{11.5}$$

gives the plane curves which are invariant under the flow φ_t, satisfying

$$\psi(0, c_1, c_2) = 0; \quad \frac{\partial \psi}{\partial x_1}(0, c_1, c_2) = 0.$$

Notice that $c_1 < 0 \Leftrightarrow x_1 < 0$ by $x_1(t) = \dfrac{c_1}{1 - c_1 t}$ for all $t \geq 0$. Therefore, when $x_1 < 0$ is sufficiently small, (11.5) is a local center manifold, which is tangent to E^c at the origin. There are infinite many center manifolds because of any $c_1 < 0$ and c_2. However, these manifolds keep the same orientation. See Fig. 11.3 as follows.

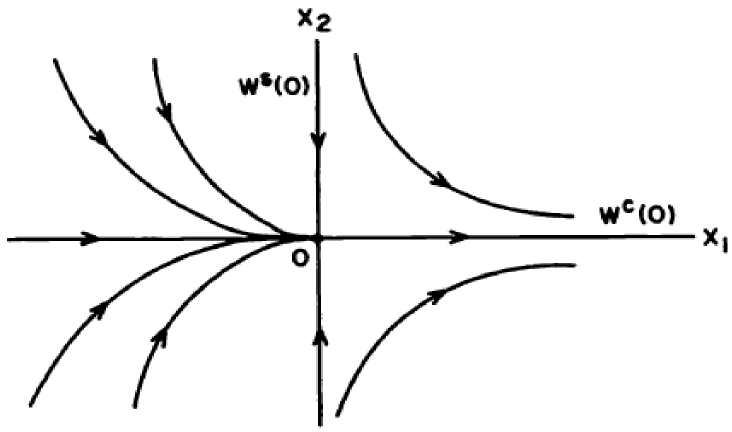

Fig. 11.3

4) Center Manifold and Reduced System

For simplicity, we assume that f in (11.1) is of C^2. Then, (11.1) can be written as

$$x' = Ax + g(x), \tag{11.6}$$

where $g(0)=0$ and $\frac{\partial g}{\partial x}(0)=0$.

Then, there exists an invertible matrix C s.t.

$$C A C^{-1} = \begin{pmatrix} A_1 & O \\ O & A_2 \end{pmatrix}.$$

where an $k\times k$ matrix A_1 has eigenvalues with zero real parts and an $(n-k)\times(n-k)$ matrix A_2 has eigenvalues with negative real parts. The change of variable

$$\begin{pmatrix} y \\ z \end{pmatrix} = C\,x\,, \quad y\in R^k\,; \quad z\in R^{n-k}$$

transform (11.6) into the form

$$y' = A_1 y + g_1(y, z)\,; \tag{11.7a}$$

$$z' = A_2 y + g_2(y, z)\,, \tag{11.7b}$$

where g_1 and g_2 inherit properties of g. They satisfy

$$g_j(0,0)=0\,;\ \ \frac{\partial g_j}{\partial y}(0,0)=0\,;\ \ \frac{\partial g_j}{\partial z}(0,0)=0 \quad \text{for} \quad j=1,2\,.$$

Definition 11.3 If $z=h(y)$ is an invariant manifold for (11.7) and h is of C^1, then it is said **a center manifold** if

$$h(0)=0 \quad \text{and} \quad \frac{\partial h}{\partial y}(0)=0\,.$$

Then, by the definition 11.3, The local center manifold, tangent to E^c at $x=0$, is given by

$$W^c(0)=\{(y,z)\in R^k\times R^{n-k}\mid z=h(y),\ \|y\|\le\delta\}\,.$$

Theorem 11.3 For (11.7), there exist $\delta>0$ and $h(y)$ of C^1, defined for all $\|y\|<\delta$ such that $z=h(y)$ is a center manifold for (11.7).

The flow on the center manifold with $z(0)=h(y(0))$ satisfies the following equation

$$y' = A_1 y + g_1(y, h(y)), \tag{11.8}$$

which we refer to as **the reduced system**. However, if $h(y)$ is unknown, (11.8) is also unknown.

On the other hand, the change of variables

$$\begin{pmatrix} y \\ w \end{pmatrix} = \begin{pmatrix} y \\ z - h(y) \end{pmatrix}$$

Transforms (11.6) into

$$y' = A_1 y + g_1(y, w + h(y)); \tag{11.9a}$$

$$w' = A_2[w + h(y)] + g_2[y, w + h(y)] - \frac{\partial h}{\partial y}(y)[A_1 y + g_1(y, w + h(y))]. \tag{11.9b}$$

In the new coordinates, the center manifold is $w = 0$. The flow on the center manifold is characterized by

$$w(t) \equiv 0 \quad \Rightarrow \quad w'(t) \equiv 0 \quad \text{for all} \quad t \geq 0.$$

Substituting $w(t) \equiv 0$ and $w'(t) \equiv 0$ into (11.9b) results in

$$0 = A_2 h(y) + g_2(y, h(y)) - \frac{\partial h}{\partial y}(y)[A_1 y + g_1(y, h(y))], \tag{11.10}$$

which is a center manifold equation for $h(y)$ must satisfy, with boundary conditions

$$h(0) = 0; \quad \frac{\partial h}{\partial x}(0) = 0. \tag{11.11}$$

Denote

$$N(h(y)) = \frac{\partial h}{\partial y}(y)[A_1 y + g_1(y, h(y))] - A_2 h(y) - g_2(y, h(y)),$$

and (11.10) becomes $N(h(y)) = 0$.

Remark 11.7 (11.10) for $h(y)$ can't be solved in most cases (to do so would imply that (11.7) can be solved), but its solution can be approximated arbitrarily closely as a Taylor series in y.

Theorem 11.4 If $\varphi(y)$ is of C^1 with $\varphi(0)=0$ and $\frac{\partial\varphi}{\partial y}(0)=0$ can be found such that $N(\varphi(y)) = O(\| y \|^p)$ for some $p>1$, then for $\| y \| << 1$,

$$h(y) - \varphi(y) = O(\| y \|^p)$$

and the reduced system (11.7) can be represented as

$$y' = A_1 y + g_1(y, \varphi(y)) + O(\| y \|^{p+1}).$$

Remark 11.8 The order of magnitude notation $O(\cdot)$, it is enough to think of $\rho(y) = O(\| y \|^p)$ as $\| \rho(y) \| \le k \| y \|^p$ for $\| y \| << 1$.

5) Stability of Center Manifold

When E^u is nontrivial, it is definitely unstable by Lyapunov stability. When E^u is trivial, the stability of the full system (11.7) or (11.6) is determined by the reduced system (11.8).

Theorem 11.5 If $y=0$ of the reduced system (11.8) is asymptotically stable (respectively unstable), then $x=0$ of the full system (11.7), or (11.6) is also asymptotically stable (respectively unstable).

Corollary 11.6 Under the conditions of Theorem 11.5, if $y=0$ of the reduced system (11.8) is stable and there is a Lyapunov function $V(y)$ of C^1 such that

$$\frac{\partial V}{\partial y}[A_1 y + g_1(y, h(y))] \le 0,$$

near $y=0$, then, $x=0$ of the full system (11.7), or (11.6) is also stable.

6) Examples

Example 11.3 Consider

$$\begin{cases} x_1' = x_2 \\ x_2' = -x_2 + a x_1^2 + b x_1 x_2 \end{cases}, \tag{11.12}$$

where $a \neq 0$.

The system has a unique equilibrium $x = 0$. The linearization at $x = 0$ results in

$$A = Df(0) = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \text{ and } g(x) = \begin{pmatrix} 0 \\ a x_1^2 + b x_1 x_2 \end{pmatrix},$$

where A has eigenvalues at 0 and -1, and $g(x)$ satisfies $g(0) = 0$ and $\frac{\partial g}{\partial x}(0) = 0$. Let M be a matrix whose columns are the eigenvectors of A; that is,

$$M = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix},$$

and take $C = M^{-1}$. Then,

$$C A C^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}.$$

The change of variables

$$\begin{pmatrix} y \\ z \end{pmatrix} = C \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ -x_2 \end{pmatrix}$$

puts the system (11.12) into the form

$$y' = a(y+z)^2 - b(yz + z^2);$$

$$z' = -z - a(y+z)^2 + b(yz + z^2).$$

The center manifold equation (11.10) with the boundary condition (11.11) becomes

$$N(h(y)) = h'(y)[a(y + h(y))^2 - b(yh(y) + h^2(y))]$$

$$+ h(y) + a(y + h(y))^2 - b(yh(y) + h^2(y)) = 0;$$

$$h(0) = h'(0) = 0.$$

We set $h(y) = h_2 y^2 + h_3 y^3 + \cdots$ and substitute it in the center manifold equation to find unknown coefficients $h_2, h_3, \cdots$ by matching coefficients of like powers in y. We don't know in advance how many terms of series we need. Let us start with the simplest approximation $h(y) \approx 0$. The reduced system is

$$y' = a y^2 + O(|y|^3).$$

The term $a y^2$ is the dominant term when $|y| << 1$. Since $a \neq 0$, the reduced system is unstable. By Theorem 11.5, the full system is unstable.

Remark 11.9 Notice that an $O(|y|^2)$ error in $h(y)$ results in an $O(|y|^3)$ error in the right-hand side of the reduced system. This is a consequence of the fact that $g_1(y,z)$, which appears on the right-hand side of the reduced system as $g_1(y,h(y))$, has a partial derivative with respect to z that vanishes at the origin. Clearly, this observation is also valid for higher order approximations.

Example 11.4 Consider the system

$$y' = yz\,; \quad z' = -z + ay^2\,.$$

The center manifold equation (11.10) with the boundary condition (11.11) is

$$N(h(y)) = h'(y)[y\,h(y)] + h(y) - ay^2 = 0\,; \quad h(0) = h'(0) = 0\,.$$

We start by trying $\varphi(y) = 0$. The reduced system is $y' = O(|y|^3)$, for which, we can't reach any conclusion about the stability of the origin.

Then, we try $h(y) = h_2 y^2 + O(|y|^4)$ and substitute it into the center manifold equation and calculate h_2, by matching coefficients of y^2, to obtain $h_2 = a$. The reduced system is

$$y' = ay^3 + O(|y|^4)\,.$$

Therefore, the origin is asymptotically stable if $a < 0$ and unstable if $a > 0$. Consequently, by Theorem 11.5, the full system is asymptotically stable if $a < 0$ and unstable if $a > 0$. If $a = 0$, the center manifold equation (11.10) with the boundary condition (11.11) reduces to

$$N(h(y)) = h'(y)[y\,h(y)] + h(y) = 0\,, \quad h(0) = h'(0) = 0\,,$$

which has the exact solution $h(y) = 0$. The reduced system $y' = 0$ is stable with $V(y) = y^2$ as a Lyapunov function. Therefore, by Corollary 11.6, the origin of the full system is stable if $a = 0$.

7) General Center Manifold Theorem

Theorem 11.7 (Center Manifold Theorem) Suppose that A in (11.2) has k eigenvalues with negative real part, j eigenvalues with positive real part and $n-k-j$ eigenvalues with zero real part. Then, there exist

1. an $m = n - k - j$ -dimensional center manifold $W^c(0)$ of class C^1 for (11.1) with $\dim W^c(0) = \dim E^c$, tangent to the center subspace E^c at $x = 0$, which is invariant under the flow φ_t of (11.1);
2. an k -dimensional stable manifold $W^s(0)$ of class C^1 for (11.1) with $\dim W^s(0) = \dim E^s$, tangent to the stable subspace E^s at $x = 0$, which is invariant under the flow φ_t of (11.1) and φ_t that starts on $W^s(0)$ is exponentially decay as $t \to +\infty$;
3. an j -dimensional unstable manifold $W^u(0)$ of class C^1 for (11.1) with $\dim W^u(0) = \dim E^u$, tangent to the unstable subspace E^u at $x = 0$, which is invariant under the flow φ_t of (11.1) and φ_t that starts on $W^u(0)$ is exponentially decay as $t \to -\infty$.

Remark 11.10 Theorem 11.2 is a general form of the center manifold theorem. It is local. The proof of the center manifold theorem is a bit harder and tedious. We will not give the proof. It may consult the textbook of “Elements of Differentiable Dynamics and Bifurcation Theory” Academic Press, New York, 1989 at p.32, by D. Ruelle.

Remark 11.11 The treatment of the case where E^u is trivial can be generalized to the case where E^u is nontrivial. It is omitted.

4. Summary

- Although the stable manifold theorem and the linearization characterize that $x' = f(x)$ and $x' = Df(0)x$ have the same stability property near a hyperbolic equilibrium, the stable manifold theorem gives much more information on geometric structures.

- The stable manifold theorem uses a geometric way to characterize the local property near a hyperbolic equilibrium. The linearization uses an analytical way to characterize the local property near a hyperbolic equilibrium.

- The center manifold approach treats the case where equilibrium is not hyperbolic, but it is much complicated.

Homework

11.1 Consider the system given by

$$x_1' = -x_1^3, \quad x_2' = -x_2 .$$

1) Find E^s and E^c;
2) Show that for any any c_1 and c_2, the function

$$x_2 = \psi(x_1, c_1, c_2) = \begin{cases} -c_1 e^{-\frac{1}{2}x_1^2}, & x_1 < 0 \\ 0, & x_1 = 0 \\ c_2 e^{-\frac{1}{2}x_1^2} & x_1 > 0 \end{cases}$$

gives one dimensional invariant manifold, which satisfy

$$\psi(0, c_1, c_2) = 0; \quad \frac{\partial \psi(0, c_1, c_2)}{\partial x_1} = 0 .$$

3) For $|x_1| << 1$, this invariant manifold is local center manifold. Show that there are infinite many local center manifolds at the origin.

11.2 Study the stability of the following system

$$y' = y^2 z - y^2; \quad z' = -z + y^2$$

by the center manifold approach.

Remark 11.7 Theorem 11.2 is a general form of the center manifold theorem. It is local. The proof of the center manifold theorem is a bit harder and tedious. We will not give the proof. It may consult the textbook of “Elements of Differentiable Dynamics and Bifurcation Theory” Academic Press, New York, 1989 at p.32, by D. Ruelle.